2 The Probability Mass Function (PMF) for a Poisson Distributed Random Variable

If y follows a Poisson distribution, then:

$$P(y = y_i \mid \lambda) = \frac{\lambda^{y_i}}{e^{\lambda} y_i!} \quad ; \quad y = 0, 1, 2, ... \quad ; \quad \lambda > 0. \tag{1}$$

The mean and variance of this distribution is λ (this equi-dispersion property proves too restrictive for most applications). Provided an estimate for λ, this PMF may be used to infer probabilities of events (e.g. $P(y \geq 6)$).

2.1 Maximum Likelihood Estimation for the Poisson Distribution

Assuming independence of the y_i's, the joint log-likelihood is:

$$l(\lambda \mid y_i) = \sum_{i=1}^{n} (y_i \log \lambda - \lambda - \log y_i!) \tag{2}$$

Or:

$$l(\lambda \mid y_i) = \sum_{i=1}^{n} y_i \log \lambda - n\lambda - \sum_{i=1}^{n} \log y_i! \tag{3}$$

Taking the derivative of (3) with respect to λ we get:

$$\frac{\partial l}{\partial \lambda} = \frac{\sum_{i=1}^{n} y_i}{\lambda} - n \tag{4}$$

Setting (4) equal to zero for the first order condition, and solving for λ, yields:

$$\tilde{\lambda} = \bar{y} \tag{5}$$

In order to verify that (5) is the MLE for λ we take the second derivative of (3) with respect to λ:

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{\sum_{i=1}^{n} y_i}{\lambda^2} \tag{6}$$

Since (6) is negative, the log-likelihood is concave and (5) solves for the global maximum. Note that (6) is the Hessian matrix, H, however, since the Poisson distribution has only one parameter (λ) the Hessian is scalar.

2.2 The Variance of $\tilde{\lambda}$

The variance of an MLE may be found by taking the inverse of the negative of the expected Hessian matrix (the matrix of second order derivatives and cross derivatives of the log-likelihood). In the present context:

$$var(\tilde{\lambda}) = [-E(H)]^{-1} = \frac{\lambda^2}{\sum E(y_i)} = \frac{\lambda^2}{n\lambda} = \frac{\lambda}{n} \tag{7}$$

Using the *invariance* property of MLEs, an MLE for the variance of $\tilde{\lambda}$ is found by substituting $\tilde{\lambda}$ into (7):

$$\widetilde{var}(\tilde{\lambda}) = \frac{\tilde{\lambda}}{n} \tag{8}$$