Example

Let $y_1, \ldots, y_n$ be a random sample from $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. Then

$$\begin{aligned} L(\mu, \sigma^2)|\mathbf{y}) &= \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}\sigma} \exp\left[-(y_i-\mu)^2/2\sigma^2\right] \\ &= \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left[-\sum(y_i-\mu)^2/2\sigma^2\right] \end{aligned}$$

giving

$$\ell(\mu, \sigma^2) = \log L(\mu, \sigma^2|\mathbf{y}) = -n\log\sigma - \sum(y_i-\mu)^2/2\sigma^2 + \text{constant}.$$

The first partial derivatives are:

$$\begin{aligned} \frac{\partial \ell}{\partial \mu} &= \frac{2\Sigma(y_i-\mu)}{2\sigma^2} \\ \frac{\partial \ell}{\partial \sigma^2} &= \frac{\Sigma(y_i-\mu)^2}{2\sigma^4} - \frac{n}{2\sigma^2} \end{aligned}$$

Setting them to zero,

$$\sum y_i - n\,\mu = 0$$

and

$$\sum(y_i-\mu)^2/2\sigma^4 - n/2\sigma^2 = 0,$$

give the m.l.e.'s $\hat{\mu} = \bar{y}$, and $\hat{\sigma}^2 = \sum(y_i-\bar{y})^2/n$, respectively.
The observed information matrix $I(\boldsymbol{\theta})$ is:

$$\begin{aligned} I(\boldsymbol{\theta}) &= -\left\{\frac{\partial^2 \ell}{\partial\theta_j \partial\theta_k}\right\} \\ &= -\begin{bmatrix} \frac{\Sigma(-1)}{\sigma^2} & \frac{-\Sigma(y_i-\mu)}{\sigma^4} \\ \frac{-\Sigma(y_i-\mu)}{\sigma^4} & \frac{-\Sigma(y_i-\mu)^2}{\sigma^6} + \frac{n}{2\sigma^4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{n}{\sigma^2} & \frac{\Sigma(y_i-\mu)}{\sigma^4} \\ \frac{\Sigma(y_i-\mu)}{\sigma^4} & \frac{\Sigma(y_i-\mu)^2}{\sigma^6} - \frac{n}{2\sigma^4} \end{bmatrix} \end{aligned}$$

and the expected information matrix $\mathcal{I}(\boldsymbol{\theta})$ is:

$$\begin{aligned} \mathcal{I}(\boldsymbol{\theta}) &= \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n\sigma^2}{\sigma^6} - \frac{n}{2\sigma^4} \end{bmatrix} = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{bmatrix} \\ \mathcal{I}(\boldsymbol{\theta})^{-1} &= \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix} \end{aligned}$$